LAU
School of Arts and Sciences

LEBANESE AMERICAN UNIVERSITY
Department of Computer Science and Mathematics

MTH101 – Calculus I
Summer I -2015
Exam 2
(June 29, 2015)

NAME: Answer Key **ID:**

Duration: 75 minutes

Instructor: Ms. Liwa Sleiman

This exam is comprised of 7 problems. Answer the questions in the space provided for each problem. If more space is needed, use the back of the page. Make sure to justify all your answers.

Problem	Grade points	
I	20	
II	06	
III	06	
IV	08	
V	20	
VI	20	
VII	20	
Total	**100**	

I. (20%)
Find

a) $\sin\left(\pi - \frac{\pi}{6}\right)$

b) $\tan\left(\frac{17\pi}{3}\right)$

c) $\sec\left(\frac{19\pi}{6}\right)$

d) $\cot\left(\frac{\pi}{4} - \frac{\pi}{3}\right)$ using addition formula

e) $\csc^2\left(\frac{\pi}{8}\right)$ using double-angle / half-angle formula

2% a) $\sin\left(\pi - \frac{\pi}{6}\right) = \sin\frac{\pi}{6} = \frac{1}{2}$ (1½) (1½)

4% b) $\tan\left(17\frac{\pi}{3}\right) = \tan\left(\frac{18\pi - \pi}{3}\right) = \tan\left(6\pi - \frac{\pi}{3}\right) = \tan\left(\frac{-\pi}{3}\right)$ (½) (½) (½)

$= -\tan\frac{\pi}{3} = -\sqrt{3}$ (2) (½)

$\boxed{\text{or}}$

$\frac{\sin\left(-\frac{\pi}{3}\right)}{\cos\left(-\frac{\pi}{3}\right)} = \frac{-\sin\frac{\pi}{3}}{\cos\frac{\pi}{3}} = \frac{-\sqrt{3}/2}{1/2} = -\sqrt{3}$ (½) (½) (½) (½) (½)

4% c) $\sec\left(\frac{19\pi}{6}\right) = \sec\left(\frac{18\pi + \pi}{6}\right) = \sec\left(3\pi + \frac{\pi}{6}\right) = \sec\left(\pi + \frac{\pi}{6}\right)$ (½) (½) (½)

$= \frac{1}{\cos\left(\pi + \frac{\pi}{6}\right)} = \frac{1}{-\cos\pi/6} = \frac{-1}{\sqrt{3}/2} = \frac{-2}{\sqrt{3}} = \frac{-2\sqrt{3}}{3}$ (1) (1) (½)

6% d) $\cot\left(\frac{\pi}{4} - \frac{\pi}{3}\right) = \frac{\cos\left(\frac{\pi}{4} - \frac{\pi}{3}\right)}{\sin\left(\frac{\pi}{4} - \frac{\pi}{3}\right)} = \frac{\cos\frac{\pi}{4}\cos\frac{\pi}{3} + \sin\frac{\pi}{4}\sin\frac{\pi}{3}}{\sin\frac{\pi}{4}\cos\frac{\pi}{3} - \cos\frac{\pi}{4}\sin\frac{\pi}{3}}$ (1) (2) (2)

$= \frac{\frac{\sqrt{2}}{2}\cdot\frac{1}{2} + \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}}{\frac{\sqrt{2}}{2}\cdot\frac{1}{2} - \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}} = \frac{\frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4}}{\frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4}} = \frac{\sqrt{2} + \sqrt{6}}{\sqrt{2} - \sqrt{6}} = -2 - \sqrt{3}$ (1)

4% e) $\csc^2\frac{\pi}{8} = \frac{1}{\sin^2\frac{\pi}{8}} = \frac{1}{\frac{1 - \cos\frac{2\pi}{8}}{2}} = \frac{2}{1 - \cos\frac{\pi}{4}} = \frac{2}{1 - \frac{\sqrt{2}}{2}} = \frac{2}{\frac{2 - \sqrt{2}}{2}} = \frac{4}{2 - \sqrt{2}}$ (1) (2) (½) (½)

$= 4 + 2\sqrt{2}$

II. **(6%)**
Show that $y = 2x - 2$ is an oblique asymptote to

$$f(x) = \frac{2x^2 + 4x + 9}{x + 3}$$

$$\lim_{x\to\infty}\left[f(x) - y\right] = \lim_{x\to\infty}\left[\frac{2x^2+4x+9}{x+3} - (2x-2)\right] \; 1$$

(1)

$$= \lim_{x\to\infty} \frac{2x^2+4x+9-(2x-2)(x+3)}{x+3} \; 1$$

$$= \lim_{x\to\infty} \frac{\cancel{2x^2}+\cancel{4x}+9-\cancel{2x^2}-\cancel{6x}+\cancel{2x}+6}{x+3} \; 1$$

$$= \lim_{x\to\infty} \frac{15}{x+3} \; 1$$

$$= \frac{15}{\infty} = 0 \; \tfrac{1}{2} \quad \Longrightarrow \quad \boxed{y = 2x-2 \text{ OA}} \; \tfrac{1}{2}$$

III. Suppose that

$$x^2 - 2x + 3 \le f(x) \le 5 - x \qquad \text{for} \quad -1 \le x \le 2.$$

Find (if possible) or bound the following limits:

a) **(2%)** $\lim_{x\to 1} f(x)$

b) **(2%)** $\lim_{x\to 2} f(x)$

c) **(2%)** $\lim_{x\to 3} f(x)$

a) using Sandwich Theorem

$$\lim_{x\to 1} x^2-2x+3 \le \lim_{x\to 1} f(x) \le \lim_{x\to 1} 5-x \; 1$$

$$\boxed{2 \le \lim_{x\to 1} f(x) \le 4} \; 1$$

b) using Sandwich theorem

$$\lim_{x\to 2} x^2-2x+3 \le \lim_{x\to 2} f(x) \le \lim_{x\to 2} 5-x \; \tfrac{1}{2}$$

$$3 \le \lim_{x\to 2} f(x) \le 3 \; \tfrac{1}{2}$$

$$\Longrightarrow \boxed{\lim_{x\to 2} f(x) = 3} \; 1$$

c) $3 \notin [-1,2]$ therefore Sandwich Theorem cannot be used! 2

IV. **(8%)** Use the Intermediate Value Theorem to prove that $x^3 + 2x^2 - 5x = -1$ has a solution between 0 and 1.

$$x^3 + 2x^2 - 5x = -1$$

Let $f(x) = x^3 + 2x^2 - 5x$ 1

$f(0) = 0$ $1\frac{1}{2}$

$f(1) = -2$ $1\frac{1}{2}$

since $y_0 = -1$ is a value between $f(0)$ and $f(1)$ 1

and since $f(x)$ is continuous 1

Then, by the Intermediate Value Theorem 2

$\exists\, c \in (0,1)$ such that $f(c) = y_0 = -1$.

V. **(20%)**
Find all the asymptotes. Show your work.

a) $f(x) = \frac{3x-5}{2x+6}$

b) $g(x) = \frac{x^3+6x-3}{x^2+3x+2}$

7% a)

$2x+6 \neq 0$

$x \neq -3$

$D = (-\infty, -3) \cup (-3, +\infty)$ 1

1 $\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} \frac{3x}{2x}$ 1 $= \frac{3}{2}$ ½ $\Rightarrow \boxed{y = \frac{3}{2} \text{ HA}}$ 1

1 $\lim_{x \to -3} f(x) = \frac{-9-5}{0} = \frac{-14}{0} = \infty$ ½ $\Rightarrow \boxed{x = -3 \text{ VA}}$ 1

13% b)

$x^2+3x+2 \neq 0$

$(x+1)(x+2) \neq 0$

$\boxed{x \neq -1}$ and $\boxed{x \neq -2}$

$D = (-\infty, -2) \cup (-2, -1) \cup (-1, +\infty)$ 1

1 $\lim_{x \to -2} g(x) = \frac{-8-12-3}{0} = \frac{-23}{0} = \infty$ ½ $\Rightarrow \boxed{x = -2 \text{ VA}}$ 1

1 $\lim_{x \to -1} g(x) = \frac{-1-6-3}{0} = \frac{-10}{0} = \infty$ ½ $\Rightarrow \boxed{x = -1 \text{ VA}}$ 1

1 $\lim_{x \to \pm\infty} g(x) = \lim_{x \to \pm\infty} \frac{x^3}{x^2}$ 1 $= \lim_{x \to \pm\infty} x$ ½ $= \pm\infty$ ½

$$
\begin{array}{r|l}
 & x - 3 \\
x^2+3x+2 & x^3 + 6x - 3 \\
 & -(x^3+3x^2+2x) \\
 & -3x^2+4x-3 \\
 & -(-3x^2-9x-6) \\
 & 13x+3
\end{array}
$$

2

$g(x) = x - 3 + \frac{13x+3}{x^2+3x+2}$ 1

$\boxed{y = x-3 \text{ OA}}$ 1

VI. **(20%)**
Compute the following limits:

2% **a)** $\lim_{x\to 1} \frac{x^2+2x-5}{2x-3} = \frac{1+2-5}{2-3} = \frac{-2}{-1} = 2$ (1) (1)

4% **b)** $\lim_{x\to -1} \frac{x^2+4x+3}{x^2+3x+2} = 0$

$= \lim_{x\to -1} \frac{(x+3)\cancel{(x+1)}}{(x+2)\cancel{(x+1)}} = \lim_{x\to -1} \frac{x+3}{x+2} = \frac{-1+3}{-1+2} = \frac{2}{1} = 2$ (2) (1) (1/2) (1/2)

5% **c)** $\lim_{x\to 5} \frac{2x^2-50}{\sqrt{2x+6}-4} = \frac{0}{0}$

$= \lim_{x\to 5} \frac{2x^2-50}{\sqrt{2x+6}-4} \cdot \frac{\sqrt{2x+6}+4}{\sqrt{2x+6}+4}$ (1)

$= \lim_{x\to 5} \frac{(2x^2-50)(\sqrt{2x+6}+4)}{2x+6-16}$ (1)

$= \lim_{x\to 5} \frac{2(x^2-25)(\sqrt{2x+6}+4)}{2x-10}$

(2) $= \lim_{x\to 5} \frac{\cancel{2(x-5)}(x+5)(\sqrt{2x+6}+4)}{\cancel{2(x-5)}}$

(1/2) $= \lim_{x\to 5} (x+5)(\sqrt{2x+6}+4)$

$= (10)(8)$

(1/2) $= 80$

6% **d)** $\lim_{x\to 0} \frac{(\tan \pi x)(\csc 2x)}{(\sin 3x)(\cot 5x)} = \frac{0}{0}$

$= \lim_{x\to 0} \tan(\pi x) \cdot \csc(2x) \cdot \frac{1}{\sin 3x} \cdot \frac{1}{\cot 5x}$ (1)

$= \lim_{x\to 0} \frac{\sin(\pi x)}{\cos(\pi x)} \cdot \frac{1}{\sin(2x)} \cdot \frac{1}{\sin 3x} \cdot \frac{\sin 5x}{\cos 5x}$ (1/2) (1/2) (1/2)

$= \lim_{x\to 0} \frac{\pi x}{\pi x}\underbrace{\frac{\sin(\pi x)}{\cos(\pi x)}}_{=1} \cdot \frac{1}{2x\underbrace{\frac{\sin(2x)}{2x}}_{=1}} \cdot \frac{1}{3x\underbrace{\frac{\sin(3x)}{3x}}_{=1}} \cdot \frac{5x}{5x}\underbrace{\frac{\sin(5x)}{\cos(5x)}}_{=1}$ (1)

$= \lim_{x\to 0} \frac{\pi \cancel{x}}{\cos(\pi x)} \cdot \frac{1}{2\cancel{x}} \cdot \frac{1}{3\cancel{x}} \cdot \frac{5\cancel{x}}{\cos(5x)}$ (1)

$= \lim_{x\to 0} \frac{5\pi}{6\,\cancel{\cos(\pi x)} \cdot \cancel{\cos(5x)}} = \frac{5\pi}{6}$ (1) (1/2)

3% **e)** $\lim_{x\to\infty} \frac{-5x^2+3x}{2x^3+7x+4}$

$= \lim_{x\to\infty} \frac{-5x^2}{2x^3}$ (1)

$= \lim_{x\to\infty} \frac{-5}{2x}$ (1)

$= \frac{-5}{\infty}$ (1/2)

$= 0$ (1/2)

VII. **(20%)**

Consider the piece-wise function

$$f(x) = \begin{cases} 6a+3 & 1 \le x \le 2 \\ a-x & 2 < x < 3 \\ b+2 & 3 \le x < 4 \\ 5 & x = 4 \\ 2x-3 & x > 4 \end{cases}$$

a) Determine a condition for "a" so that $\lim_{x\to 2} f(x)$ exists.

b) Determine a condition for "b" so that f(x) is continuous at x=4.

c) Can f(x) be continuous everywhere ? Justify your answer.

7% a) $\lim_{x\to 2} f(x)$ exists $\Rightarrow$ 1 $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x)$ 1

1 $= \lim_{x\to 2^-} 6a+3$ | $= \lim_{x\to 2^+} a-x$ 1

1 $= 6a+3$ | $= a-2$ 1

$6a+3 = a-2$

$5a = -5$ 1

$\boxed{a=-1}$

8% b) f(x) is continuous at $x=4$

$\Rightarrow$ 1 $\lim_{x\to 4^-} f(x) = \lim_{x\to 4^+} f(x)$ 1 $= f(4)$ 1

1 $= \lim_{x\to 4^-} b+2$ | $= \lim_{x\to 4^+} 2x-3$ 1 | $=5$

1 $= b+2$ | $= 5$ 1

$b+2=5$

$\boxed{b=+3}$ 1

5% c) for $a=-1$ and $b=3$ f(x) cannot be continuous everywhere since f(x) is not continuous at $x=3$ 1

1 $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} -1-x = -1-3 = -4$ $\frac{1}{2}$

1 $\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} 3+2 = 5$ $\frac{1}{2}$

1 $f(3) = 5$